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On the Møller's energy complex of the charged dilaton black hole

I-Ching Yang ¹, Wei-Fui Lin ², and Rue-Ron Hsu ³

Department of Physics, National Cheng Kung University
Tainan, Taiwan 701, Republic of China

ABSTRACT

Using Møller's energy complex , we obtain the energy distributions of GHS solution and dyonic dilaton black hole solution in the dilaton gravity theory. It is confirmed that the Møller's energy complex is indeed a 3-scalar under purely spatial transformation in these energy distributions. Some interested properties of the energy distribution of dyonic black hole are discussed.

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¹E-mail:icyang@ibm65.phys.ncku.edu.tw

²E-mail:wfl@ibm65.phys.ncku.edu.tw

³E-mail:rrhsu@mail.ncku.edu.tw

The well - known Einstein's energy complex is the foremost definition of energy complex. This idea comes from that the continuity equation $\frac{\partial T^\mu_\nu}{\partial x^\mu} = 0$ can give out a conserved quantity in the absence of a gravitation field. However, when the gravitational field is present, the conservation law will be generalized to be

$$\nabla_\mu T^\mu_\nu = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} T^\mu_\nu)}{\partial x^\mu} - \frac{1}{2} \frac{\partial g_{\mu\sigma}}{\partial x^\nu} T^{\mu\sigma} = 0. \quad (1)$$

It does not generally induce any conservation quantity. In order to determine the conserved total four-momentum, a particular coordinate system where all the first derivatives of the $g_{\mu\nu}$ are vanish at some particular points must be chosen. However, the Einstein's energy complex does not supply a physically satisfactory description of the energy distribution or of the energy contained in a limited part of space. Due to the condition that the first - order derivative of $g_{\mu\nu}$ must be equal to zero, one must specify the calculation at the large spatial distance r from the system in Cartesian coordinate. Thus, it has no meaning to speak of a definite localization of gravitation field in this case [1]. Another energy complex which has the property of transforming as a 3 - scalar density with respect to the group of purely spatial transformations was defined by Møller [2, 3]. This property was shown that the energy distribution will be the same while calculating in different kinds of spatial coordinate (but the same time coordinate). In this paper, we calculate the energy distributions of two black hole solutions in dilaton gravity theory to examine the property of Møller energy complex.

In the dilaton gravity theory in which the gravity is coupled to the electromagnetic and dilaton fields can be described by the four - dimensional effective string action [4]. The action can be expressed as

$$I = \int d^4x \sqrt{-g} \left[-R + 2(\nabla\phi)^2 + e^{-2\phi} F^2 \right]. \quad (2)$$

Grafinkle, Horowitz and Strominger found a charged dilaton black hole solutions (GHS solutions) [4] in a peculiar coordinate form. Their solutions are

$$ds^2 = \left(1 - \frac{2M}{\tilde{r}}\right) dt^2 - \frac{1}{\left(1 - \frac{2M}{\tilde{r}}\right)} d\tilde{r}^2 - \left(1 - \frac{\alpha}{\tilde{r}}\right) \tilde{r}^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (3)$$

$$\alpha = \frac{Q_e^2}{M} e^{2\phi_0}, \quad (4)$$

$$e^{-2\phi} = e^{-2\phi_0} \left(1 - \frac{Q_e^2}{M \tilde{r}} e^{2\phi_0}\right), \quad (5)$$

$$F_{01} = \frac{Q_e}{\tilde{r}^2} e^{2\phi}. \quad (6)$$

The properties of the GHS solutions are characterized by the mass M , electric charge Q_e and asymptotic value of the dilaton ϕ_0 . On the other hand, by using a standard spherical coordinate form

$$ds^2 = \Delta^2 dt^2 - \frac{\sigma^2}{\Delta^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \quad (7)$$

Cheng, Lin and Hsu [5] had obtained the dyonic dilaton black hole solutions (CLH solutions)

$$\Delta^2 = 1 - \frac{2M}{r^2} \sqrt{r^2 + \lambda^2} + \frac{\beta}{r^2}, \quad (8)$$

$$\sigma^2 = \frac{r^2}{r^2 + \lambda^2}, \quad (9)$$

$$\lambda = \frac{1}{2M}(Q_e^2 e^{2\phi_0} - Q_m^2 e^{-2\phi_0}), \quad (10)$$

$$\beta = (Q_e^2 e^{2\phi_0} + Q_m^2 e^{-2\phi_0}), \quad (11)$$

$$e^{2\phi} = e^{2\phi_0} \left(1 - \frac{2\lambda}{\sqrt{r^2 + \lambda^2} + \lambda}\right), \quad (12)$$

$$F_{01} = \frac{Q_e}{r^2} e^{2\phi}, \quad (13)$$

$$F_{23} = \frac{Q_m}{r^2}. \quad (14)$$

The properties of the CLH solutions are characterized by the mass M , electric charge Q_e , magnetic charge Q_m and asymptotic value of the dilaton ϕ_0 . These solutions are related to the GHS solutions by a coordinate transformation

$$\tilde{r} = \sqrt{r^2 + \frac{Q_e^4}{4M^2} e^{4\phi_0}} + \frac{Q_e^2}{2M} e^{2\phi_0}, \quad (15)$$

as the magnetic charge Q_m is set to be zero. This new coordinate remove the peculiar singularity $\tilde{r} = \frac{Q_e^2}{M} e^{2\phi_0}$ whose area is zero, to the essential singularity $r = 0$.

Recently, the energy distribution according to the Einstein's energy - momentum pseudotensor was studied. Based on the GHS solutions, Virbhadra et. al. [6] found a charge independent result

$$E(r) = M, \quad (16)$$

in which the positive energy is confined to the interior of the black hole. On the other hand, based on the CLH solutions, we obtained a charge dependent

result [7] which is different from Virbhadra's

$$E(r) = M + \frac{M\lambda^2}{r^2} - \frac{1}{2\sqrt{r^2 + \lambda^2}} \left(\frac{\beta\lambda^2}{r^2} + \lambda^2 + \beta \right), \quad (17)$$

By comparing Eq.(16) and (17), we find that the different coordinates chosen will induce the same total energy but different energy distributions without any relation. This shortcoming can be overcome by using Møller's energy - momentum pseudotensor.

Møller's energy - momentum pseudotensor [2] is

$$\Theta_\nu^\mu = \frac{1}{8\pi} \frac{\partial \chi_\nu^{\mu\sigma}}{\partial x^\sigma}, \quad (18)$$

where

$$\chi_\nu^{\mu\sigma} = \sqrt{-g} \left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta} - \frac{\partial g_{\nu\beta}}{\partial x^\alpha} \right) g^{\mu\beta} g^{\sigma\alpha}. \quad (19)$$

The Greek indices run from 0 to 3 and x^0 is time coordinate. Then, the energy component E is given by

$$\begin{aligned} E &= \int \int \int \Theta_0^0 dx^1 dx^2 dx^3 \\ &= \frac{1}{8\pi} \int \int \int \frac{\partial \chi_0^{0k}}{\partial x^k} dx^1 dx^2 dx^3, \end{aligned} \quad (20)$$

where the Latin index takes values from 1 to 3. The Møller's energy - momentum pseudotensor which differ from Einstein's energy - momentum pseudotensor is not necessary to carry out the calculation in the quasi - Cartesian coordinate, so we can calculate in the spherical coordinate.

In the case of the GHS solutions, we obtain the nonvanishing components χ_0^{0k} in Eq.(20),

$$\chi_0^{01} = (2M - \frac{2M\alpha}{\tilde{r}})\sin\theta. \quad (21)$$

Applying the Gauss theorem, and plugging (21) into (20), we evaluate the integral over the surface of a sphere with radius r .

$$E(r) = \frac{1}{8\pi} \oint_r (2M - \frac{2M\alpha}{r})\sin\theta d\theta d\varphi. \quad (22)$$

Finally, for the GHS solution, we find that the energy within a sphere with radius r is

$$E(r) = M - \frac{Q_e^2}{\tilde{r}} e^{2\phi_0}. \quad (23)$$

In the case of the CLH solutions, the nonvanishing components χ_0^{0k} in (19) are

$$\chi_0^{01} = (2M + \frac{4M\lambda^2}{r^2} - \frac{2\beta}{r^2}\sqrt{r^2 + \lambda^2})\sin\theta. \quad (24)$$

For the CLH solution, the energy within a sphere with radius r is

$$E(r) = M + \frac{2M\lambda^2}{r^2} - \frac{\beta}{r^2}\sqrt{r^2 + \lambda^2}. \quad (25)$$

The energy distributions are shared both by the interior and by the exterior of those charged dilaton black hole. We plot the energy distributions of the dyonic black holes or the extremal dyonic black holes by "GNUPLOT". For the dyonic black hole or the extremal dyonic black hole, see Fig.1 and Fig.4, we find that the energy distribution can be positive or negative, but they are both positive in the region $r > r_H$. For the pure electric or pure

magnetic charged black hole, i.e. $Q_m = 0$ or $Q_e = 0$, we find the remarkable property that the energy distributions are always positive except at singular point $r = 0$, see Fig.2,3,5,6. These results indicate that the physical charged black hole solution is either pure electric or pure magnetic when the positive definite condition, all the energy distribution function are positive definite except the singularity, is imposed.

Comparing the Møller' energy distributions of the GHS solutions and the CLH solutions, we find that the results of the CHL solutions seem to be different from the GHS solutions. But they are related by scratching the magnetic charge Q_m of CHL solution and by the coordinate transformation (15) which is a purely spatial transformation. Therefore, it will be the same energy distributions of the CLH solutions and the GHS solutions. Then we confirm that the statement in Møller's paper, "the property of the Møller's energy complex is that transforms as 3 - scalar with respect to the group of purely spatial transformation", is still valid for the dilaton gravity theory.

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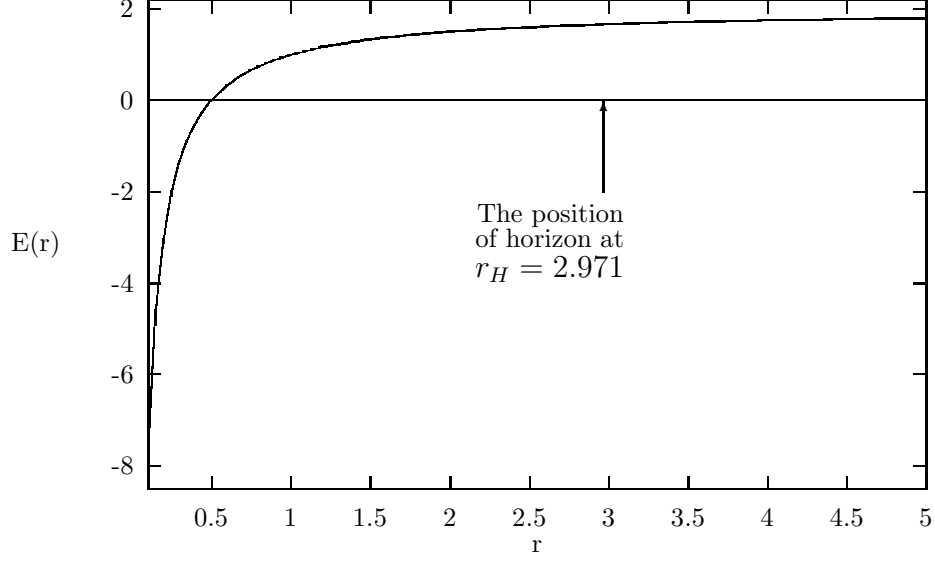


Figure 1: The energy distribution of dyonic black hole with $\phi_0 = 0$, $M = 2$, $Q_e = 1$ and $Q_m = 1$.

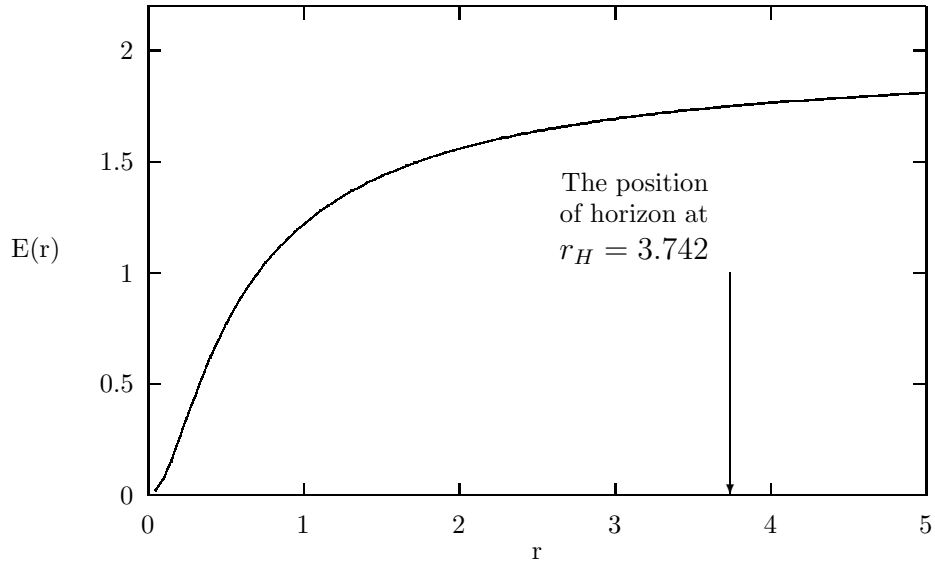


Figure 2: The energy distribution of pure electric black hole with $\phi_0 = 0$, $M = 2$, $Q_e = 1$ and $Q_m = 0$.

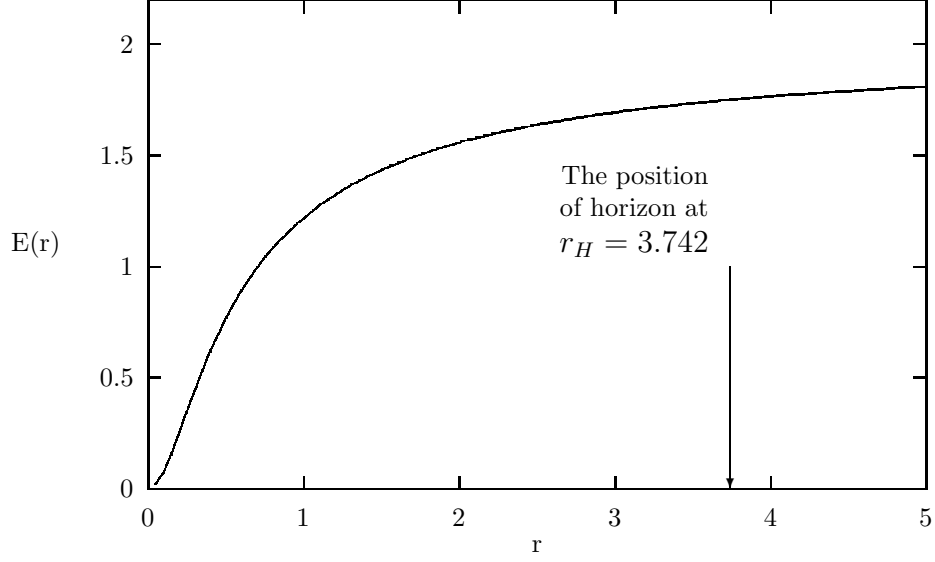


Figure 3: The energy distribution of pure magnetic black hole with $\phi_0 = 0$, $M = 2$, $Q_e = 0$ and $Q_m = 1$.

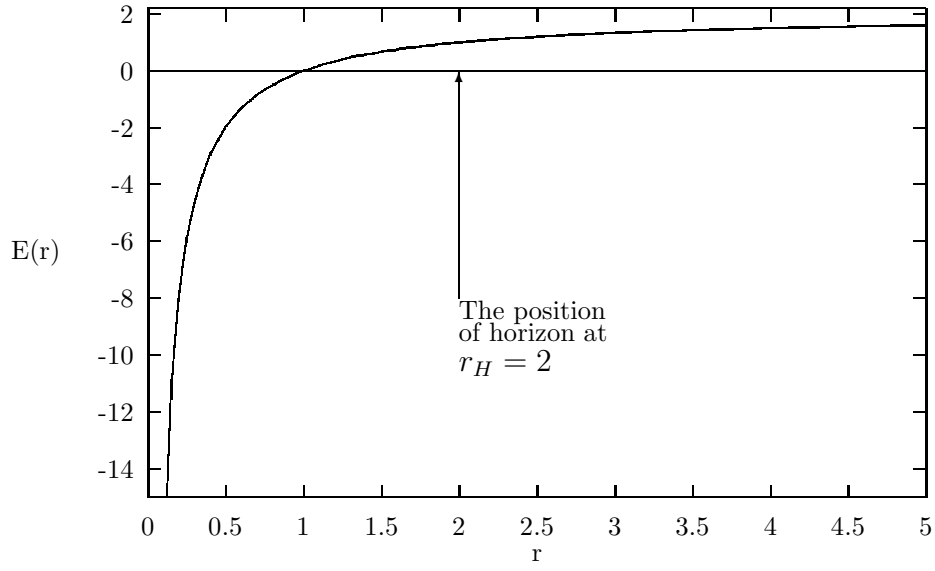


Figure 4: The energy distribution of extremal dyonic black hole with $\phi_0 = 0$, $Q_e = \sqrt{2}$ and $Q_m = \sqrt{2}$.

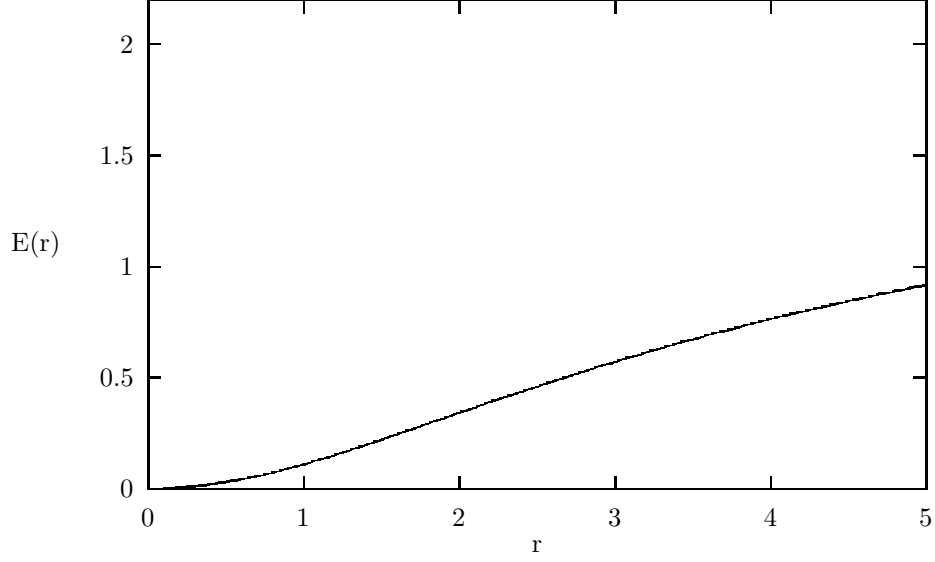


Figure 5: The energy distribution of extremal electrically charged black hole with $\phi_0 = 0$, $M = 2$, $Q_e = 0$ and $Q_m = 2\sqrt{2}$.

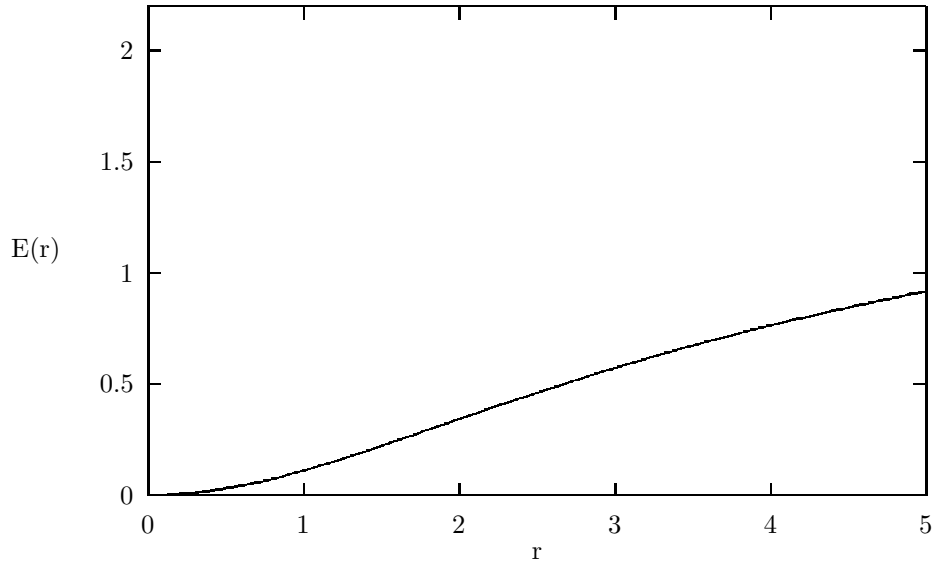


Figure 6: The energy distribution of extremal magnetically charged black hole with $\phi_0 = 0$, $M = 2$, $Q_e = 2\sqrt{2}$ and $Q_m = 0$.